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Simple Field Theoretical Models on Noncommutative Manifolds

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Abstract

We review recent progress in formulating two-dimensional models over noncommutative manifolds where the space-time coordinates enter in the formalism as non-commuting matrices. We describe the Fuzzy sphere and a way to approximate topological nontrivial configurations using matrix models. We obtain an ultraviolet cut off procedure, which respects the symmetries of the model. The treatment of spinors results from a supersymmetric formulation; our cut off procedure preserves even the supersymmetry.

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1. Introduction

We intend to use part of the ideas of noncommutative geometry [1], and apply them to simple models of quantum field theory. Simple refers here just to two dimensions. Moreover we shall treat the euclidean version of the models and compactify space time. Thus we arrive at the two-sphere S^2, although other manifolds can be treated.

Following old ideas of von Neumann, we encode as much structure as possible, within the commutative algebra of smooth functions defined over the manifold M, which we denote by $\mathcal{A}_\infty$. Next we shall approximate this algebra by a sequence of noncommutative algebras $\mathcal{A}_n$, which converge in a specific manner to $\mathcal{A}_\infty$ for $n \to \infty$.

One formulates next the differential geometry on the manifold M in terms of operations on $\mathcal{A}_\infty$. For example vector fields on M can be identified as derivations on $\mathcal{A}_\infty$. Differential forms can be defined as duals of vector fields or with the help of a Dirac operator plus some additional structure. The last formulation can be applied in the noncommutative case too, and a differential calculus over noncommutative algebras results.

The geometry so obtained, is often referred to a pointless geometry. In the commutative case, abelian ideals of the algebra correspond to points of the manifold. Our algebras will have no nontrivial ideals.

Next, fields have to be defined as sections of line, respectively, spinor bundles, or gauge field potentials as connections of principal bundles. They form modules over $\mathcal{A}_\infty$ and the natural noncommutative generalization consists in studying finitely generated projective modules, which will enter our discussion later on. Finally an integral calculus is needed, in order to integrate our forms. We will have no problems in that respect, since all our algebras will be finite dimensional. Also de Rham cohomology has been generalized to cyclic cohomology, and cyclic cocycles replace the notion of an integral.

Let us mention, for later use, a simple example: the differential calculus on matrix algebras $M_n = \mathrm{Mat}\,(n, \mathbf{C})$ [2]. Let the algebra be generated by $\{\mathbb{1}, \lambda_i\}$, $i = 1, \dots n^2 - 1$. Vector fields can be identified as derivations $e_i = \mathrm{ad}\,\lambda_i$. A difference to the calculus on manifolds shows up: derivations do not form anymore a left module.

Differential forms may be introduced by duality:

$$(d\,\lambda_j)\,(e_i) = e_i\,(\lambda_j) = C_{ijk}\,\lambda_k\,, \tag{1}$$

where C_{ijk} denote the structure constants.

Zero-forms are identified with algebra elements $\Omega^0 = M_n$; one-forms have been just defined $\Omega^1 = \{a\,db \mid a, b \in M_n\}$, and there is no difficulty in extending d, such that the full differential complex results. Knowing one-forms allows to define a covariant derivative $D = d + A$ and a curvature two-form $F = D^2 \equiv dA + A^2$. Gauge transformations can be

defined and an action functional for gauge fields can be written down. Thus one can define a gauge model over a matrix algebra. The standard steps, like writing down a volume form and a star operation, introducing $\delta = * \, d \, *$ and defining the Laplace Beltrami operator and the Lie derivative, can be done as in the commutative case.

2. The Fuzzy Sphere

Let us first give a suitable description of the set of smooth functions defined over the usual sphere $S^2 \hookrightarrow \mathbf{R}^3$. Let X_i denote three commuting coordinates $[X_i, X_j] = 0$. The algebra we are dealing with is given by

$$\mathcal{A}_\infty = \{ f(X_1, X_2, X_3) \mid f\text{-analytic} \} / I \,, \tag{2}$$

where the ideal I consists of functions vanishing at $\sum_{i=1}^{3} X_i^2 = R^2$. We are interested in models which are invariant under rotations of the sphere. The generators of rotations are given by

$$L_i = \epsilon_{ijk} \, X_j \, \frac{1}{i} \, \frac{\partial}{\partial X_k} \,, \tag{3}$$

which obey the $su(2)$ commutation relations. An invariant action corresponding to a free 2-dimensional scalar field ϕ is given by

$$S\,[\phi] = \sum_{i=1}^{3} \langle L_i \, \phi \mid L_i \, \phi \rangle \,, \tag{4}$$

where we have introduced the scalar product

$$\langle \phi \mid \psi \rangle = \int \frac{d^3 \, X}{2\pi \, R} \, \delta(\vec{X}^2 - R^2) \, \phi^+(X) \, \psi(X) \tag{5}$$

within the algebra $\mathcal{A}_\infty$.

From $SU(2)$ transformation properties, we deduce that X_i transforms according to the spin 1 irreducible representation and higher order products of X_i's transform according to higher spin representations. Our algebra can therefore be decomposed in the following way

$$\mathcal{A}_\infty = [0] \oplus [1] \oplus \cdots \oplus [j] \oplus \cdots \,, \tag{6}$$

where $[j]$ means the vector space of the spin j representation.

The *truncation* or quantization of $\mathcal{A}_\infty$ is now *defined* as the family of noncommutative algebras $\mathcal{A}_j$ given by the truncated sum of irreducible spin j representations [3]

$$\widehat{\mathcal{A}}_j = [0] \oplus [1] \oplus \cdots \oplus [j] \,, \tag{7}$$

equipped with an associate product and scalar product, which give in the limit $j \to \infty$ the standard product in $\mathcal{A}_\infty$. In order to define product we consider the space $\mathcal{L}\left(\frac{j}{2}, \frac{j}{2}\right)$ of linear operators from the representation space for spin $j/2$ to itself. $SU(2)$ acts on $\mathcal{L}\left(\frac{j}{2}, \frac{j}{2}\right)$ by the adjoint action. This representation is reducible and the standard Clebsch-Gordan decomposition implies that $\mathcal{L}\left(\frac{j}{2}, \frac{j}{2}\right) \equiv \mathcal{A}_j$. By the standard matrix multiplication in $\mathcal{L}\left(\frac{j}{2}, \frac{j}{2}\right)$ we obtain a noncommutative product in $\mathcal{A}_j$. As a scalar product, we take

$$\langle f \mid g \rangle_j = \frac{1}{j+1} \operatorname{Tr} f^+ g \ , \ \ f, g \in \mathcal{A}_j \, . \tag{8}$$

We are now able to make more precise in which way the product and scalar product so defined converge to their commutative limits. There is a natural chain of embeddings of vector spaces

$$\mathcal{A}_0 \hookrightarrow \mathcal{A}_1 \hookrightarrow \mathcal{A}_2 \hookrightarrow \ldots \mathcal{A}_j \hookrightarrow \ldots \mathcal{A}_\infty \, . \tag{9}$$

Any normalized element from $\mathcal{A}_j$ of the form $C^{(j)}_{\ell m}\, L^m_-\, X^{+\ell}_{(j)}$ is mapped to a normalized element from $\mathcal{A}_k$ given by $C^{(k)}_{\ell m}\, L^m_-\, X^{+\ell}_{(k)}$. Here $X^\alpha_{(j)}$ denote the representatives of the $su(2)$ generators in the irreducible representation with spin $j/2$. They obey the relations [4]

$$[X^m_{(j)}\, , \, X^n_{(j)}] = i\, \frac{R\, \epsilon_{mnp}}{\sqrt{\frac{j}{2}\, \left(\frac{j}{2}+1\right)}}\, X^p_{(j)}\, , \tag{10}$$

and the normalization coefficients $C^{(j)}_{\ell m}$ and $C^{(k)}_{\ell m}$ are determined through the requirement that the embedding conserves the norm $\|X^i_{(j)}\|^2_j = \frac{R^2}{3}$. We should also note that the defining relation for the surface S^2

$$\sum_{i=1}^{3} (X^i_{(j)})^2 = R^2 \tag{11}$$

follows from (10) and holds therefore in the noncommutative case, too.

According to (9), elements f, g of $\mathcal{A}_j$ can be canonically considered as elements of $\mathcal{A}_k$ with $k \geq j$. Their product in every $\mathcal{A}_k$ can be embedded in $\mathcal{A}_\infty$ too. If we denote it by $(fg)_k$ we can prove that

Lemma:

$$\lim_{k\to\infty}\ (f\, g)_k = f\, g\, , \tag{12}$$

where on the rhs. the commutative multiplication in $\mathcal{A}_\infty$ is meant. As for the proof, we have to show that the coefficients $C^{(k)}_{\ell m}$ converge to their commutative analog: $\lim\limits_{k\to\infty}\ C^{(k)}_{\ell m} = C^{\infty}_{\ell m}$, which can be done after the norm of $\|(X^+_{(k)})^\ell\|^2_k = (C^{(k)}_{\ell 0})^{-2}$ is evaluated.

We may remark that the truncated sphere was introduced also by Berezin [5], who quantized the symplectic two-form on the ordinary two-sphere. Hoppe [6] investigated properties of spherical membranes and introduced a truncation of high frequency excitations, which led him to the quantum sphere too. Motivated by noncommutative geometry, Madore has reinvented the Fuzzy sphere [4] and has provided an important change of the viewpoints: instead of understanding the non-commutative manifold as the *final* product of the deformation quantization he developed some basic differential geometry *starting* from the notion of the non-commutative manifold. Field theories on the Fuzzy sphere were formulated in [2] and [3, 7, 8, 9].

It is interesting to remark, that a different approach, the so called orbit method using coherent states leads to the same structure [10]. One starts with a group G (in our case $SU(2)$) and a unitary irreducible representation $T(g)$, $g \in G$ in a Hilbert space $\mathcal{H}$. Let $|0\rangle \in \mathcal{H}$, then $T(g)|0\rangle$ has a stability group H (in our case $U(1)$), and the set $\{T(g)|0\rangle\}$ parametrizes the homogeneous space G/H (which becomes in our case S^2). These states $\{|x\rangle \mid x \in G/H\}$ are over complete $\int d\mu(x)|x\rangle\langle x| = \mathbb{I}$ and allow to quantize $f \rightarrow \widehat{f} = \int f(x)\,|x\rangle\,\langle x|\,d\mu(x)$ and to dequantize $f(x) = \langle x|\,\widehat{f}\,|x\rangle$ functions over G/H. Convolution on the group defines the so called star-product. Applying the star product to the coordinate functions gives a noncommutative product. In case $T(g)$ is taken to be the spin j representation of $SU(2)$ the matrix algebra we have discussed before, results.

From the last remarks it becomes clear, that our procedure works not only for $SU(2)/U(1) \simeq S^2$, but more general. We did work out [10] a few more details for the factor space $SU(1,1)/U(1)$ which becomes a two-sided hyperboloid. Clearly the appropriate algebra becomes then infinite dimensional. In addition, the group $SU(1,1)$ has more than one series of unitary representations.

In the following we shall deal only with S^2 and finite matrix approximations. Physics is then reduced to finite degrees of freedom. The sphere is covered by a finite number of cells. In this way we introduce a fundamental length into the system, although our main interest concerns the fact that an ultraviolet cutoff results.

3. Scalar Field Theory

From the notions introduced above it is easy to formulate a scalar field theory. The truncated action may be taken to be

$$S_N\,[\phi^+,\phi] = \frac{1}{N+1}\,\mathrm{Tr}_N\{L_i\,\phi^+\,L_i\,\phi + \mathrm{Pol}\,(\phi^+,\phi)\}\,, \quad N = j, \tag{13}$$

where Pol(.,.) is a positive polynomial. Expectation values of observables $F(\phi^+,\phi)$ are

now given by

$$\langle F\rangle_N = \frac{1}{Z_N}\int [d\,\phi^+\,d\,\phi]_N\, e^{-S_N[\phi^+,\phi]}\, F(\phi^+,\phi)\,, \tag{14}$$

where the measure means integrating over all $(N+1)^2$ matrix elements.

We therefore obtained an ultraviolet cut-off through the use of the noncommutative space. But, in addition, the space symmetries, that is to say, rotations of the sphere, leave the action invariant, as long as the polynomial in equation (13) is rotational invariant [9]. This is one of the novel features. We approximated the quantum field by a finite number of modes, but kept all the symmetries of the model.

We note that reflection positivity is obeyed too and the Osterwalder-Schrader axioms hold.

Although we are mainly concerned with nonperturbative aspects, the Feynman rules are of interest too [9]. The free propagation is given by $\delta_{\ell\ell'}\,\delta_{mm'}\,\frac{1}{(\ell+\frac{1}{2})^2}$. For a $(\phi^+\,\phi)^2$ interaction, for example the four vertex introduces a nonlocality. Locality is obtained for $N\to\infty$, but the old difficulties of quantum field theory show up, too. The tadpole graph, for example, will diverge like $\ln N$.

We developped actually a second way to quantize fields defined over S^2 [11]. In addition to results obtained before, this procedure will allow us to deal with nontrivial topological configurations, or in other words with sections of $U(1)$-bundles over S^2 and their matrix approximations. Classical topological nontrivial configurations on $S^2 \hookrightarrow \mathbf{R}^3$ are classified with the help of the Hopf fibration. We start from $\mathbf{C}^2 \ni (\chi_1,\chi_2)$ and restrict the two complex coordinates χ_α to lie on S^3 of radius $\sqrt{R}$: $|\chi_1|^2+|\chi_2|^2=R$. Next we introduce the mapping $\chi_\alpha \to X_i = \chi^+\,\sigma_i\,\chi \in \mathbf{R}^3$. The restriction of χ_α to the three-sphere S^3 of radius $\sqrt{R}$ implies that X_i lies on $S^2 : \sum\limits_i \,(X_i)^2 = R^2$. Since X_i's do not change under the transformation

$$\chi \to e^{\frac{i}{2}\psi}\,\chi \quad , \quad \chi^+ \to e^{-\frac{i}{2}\psi}\,\chi^+\,, \tag{15}$$

we see that the fiber is $U(1)$.

As $\mathcal{A}_k$, $k \in \frac{1}{2}\,Z$, we denote the linear space of functions in $\mathbf{C}^2$ (or S^3 after the restriction) of the form

$$\mathcal{A}_k = \tag{16}$$
$$\{f(\chi^+,\chi) = \sum a_{m_1m_2n_1n_2}\,(\chi_1^+)^{m_1}\,(\chi_2^+)^{m_2}\,(\chi_1)^{n_1}\,(\chi_2)^{n_2} \mid 2k = m_1+m_2-n_1-n_2\}\,.$$

Under the transformation (15) $f\in\mathcal{A}_k$ goes into $e^{-ik\psi}\,f$. They are eigenfunctions of the operator $K_0 = \frac{1}{2}(\chi_\alpha^+\,\partial_{\chi_\alpha^+} - \chi_\alpha\,\partial_{\chi_\alpha})$ with eigenvalue k. There is an involutive gradation

with $\mathcal{A}_k^+ = \mathcal{A}_{-k}$ and $\mathcal{A}_k \, \mathcal{A}_e \subset \mathcal{A}_{k+e}$. $\mathcal{A}_0$ is the algebra of polynomials in the variables X_i and $\mathcal{A}_k$ are $\mathcal{A}_0$-modules.

The generators of rotations in S^2 can now be expressed in terms of the new variables as

$$L_i = \frac{i}{2} \, (\chi_\alpha^+ \, \sigma_{\alpha\beta}^{*i} \, \partial_{\chi_\beta^+} - \chi_\alpha \, \sigma_{\alpha\beta}^i \, \partial_{\chi_\beta}) \, . \tag{17}$$

They satisfy the $su(2)$ algebra in $\mathcal{A}_k$ and leave invariant the function defining the sphere: $L_i \sum\limits_k \, (\chi^+ \, \sigma_k \, \chi)^2 = 0$.

We shall actually work with two independent derivatives (corresponding to the restriction to the sphere) defined by

$$K_+ \, f = i \, \epsilon_{\alpha\beta} \, \chi_\alpha^+ \, \partial_{\chi_\beta} \, f \quad , \quad K_- \, f = i \, \epsilon_{\alpha\beta} \, \partial_{\chi_\alpha^+} (f) \, \chi_\beta \, . \tag{18}$$

The operators $K_+ \, , \, K_- \, , \, K_0$ satisfy $su(2)$ algebra relations. The following relation holds between Casimir operators of the representations (17) and (18)

$$\sum_i L_i^2 = K_0^2 + \frac{1}{2} \, (K_+ \, K_- + K_- \, K_+) \, . \tag{19}$$

The action of a complex field $\phi \in \mathcal{A}_k$ with topological charge $2k$ is now given in terms of the scalar product introduced in (5) as

$$S_k \, [\phi] = \langle \phi^+ \, | K_+ \, K_- + K_- \, K_+ | \, \phi \rangle + \langle 1 | V(\phi^+, \phi) \rangle \, . \tag{20}$$

We describe next the noncommutative version of the above steps using the Jordan-Schwinger realization of $su(2)$ [11]. This way we obtain also a noncommutative version of the Hopf fibration. We replace coordinates χ_α^+ and χ_α by the following combinations of creation and annihilation operators A_α^+ and A_α

$$\chi_\alpha^+ \to \frac{1}{\sqrt{\hat{N}}} \, A_\alpha^+ \quad , \quad \chi_\alpha \to \, A_\alpha \, \frac{1}{\sqrt{\hat{N}}}, \, \hat{N} = A_1^+ A_1 + A_2^+ A_2. \tag{21}$$

Here A_α^+, A_α obey the canonical commutation relations:

$$[A_\alpha \, , \, A_\beta^+] = \delta_{\alpha\beta} \quad , \quad [A_\alpha \, , \, A_\beta] = [A_\alpha^+ \, , \, A_\beta^+] = 0 \, . \tag{22}$$

We represent A_α^+ and A_α in the bosonic Fock space $\mathcal{F}$ spanned by vectors

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1! \, n_2!}} \, (A_1^+)^{n_1} \, (A_2^+)^{n_2} |0\rangle \, , \tag{23}$$

where the vacuum is defined by $A_1|0\rangle = A_2|0\rangle = 0$. The operators

$$X_k = \frac{1}{2}\, A^+_\alpha\, \sigma^k_{\alpha\beta}\, A_\beta \tag{24}$$

fulfil the $su(2)$ algebra relations in $\mathcal{F}$. In what follows, we need restrictions of $\mathcal{F}$ to the $(N+1)$-dimensional subspaces $\mathcal{F}_N \subset \mathcal{F}$,

$$\mathcal{F}_N = \{|n_1, n_2\rangle \mid n_1 + n_2 = N\} \quad , \quad N = 0, 1, 2, \ldots \tag{25}$$

on which the $su(2)$ algebra is represented irreducibly. The Casimir operator $C = X_3^2 + \frac{1}{2}(X_+ X_- + X_- X_+)$ takes the value $\frac{N}{2}\left(\frac{N}{2}+1\right)$ on $\mathcal{F}_N$. Appropriately scaled we arrive again at the noncommutative "sphere".

As $\widehat{\mathcal{A}}_{MN}$ we denote the space of linear mappings from $\mathcal{F}_N$ to $\mathcal{F}_M$ spanned by monomials $(A_1^+)^{n_1}\,(A_2^+)^{n_2}\,(A_1)^{m_1}\,(A_2)^{m_2}$ with $n_1 + n_2 \le M$, $n_1 + n_2 \le N$ and $2k = M - N = n_1 + n_2 - m_1 - m_2$. We obviously have $\widehat{\mathcal{A}}^+_{MN} = \widehat{\mathcal{A}}_{NM}$, $\widehat{\mathcal{A}}_{LM}\,\widehat{\mathcal{A}}_{MN} \subset \widehat{\mathcal{A}}_{LN}$. Operators from $\widehat{\mathcal{A}}_{MN}$ maps $\mathcal{F}_N$ to $\mathcal{F}_M$ and can be represented by $(N+1)$ times $(M+1)$ dimensional matrices. There is an antilinear isomorphism between $\widehat{\mathcal{A}}_{MN}$ and $\widehat{\mathcal{A}}_{NM}$ given by hermitian conjugation of these matrices. In particular $\widehat{\mathcal{A}}_{NN}$ is the $(N+1)^2$-dimensional algebra of $(N+1)\times(N+1)$ square matrices. Obviously, $\widehat{\mathcal{A}}_{MN}$ is a left $\widehat{\mathcal{A}}_{MM}$ – and a right $\widehat{\mathcal{A}}_{NN}$ module.

Generators L^j act in $\widehat{\mathcal{A}}_{MN}$ as

$$L^j\, f = X^j_{(M)}\, f - f\, X^j_{(N)}\,, \tag{26}$$

where $X^j_{(N)}$ denotes the representation of the operator X^j in $\mathcal{F}_N$. This $su(2)$ representation is reducible and equivalent to the direct product of two $su(2)$ representations

$$\left[\frac{M}{2}\right] \otimes \left[\frac{N}{2}\right] = [|k|] \oplus [|k|+1] \oplus \ldots \oplus \left[\frac{M+N}{2}\right], \tag{27}$$

where $2k = M - N$. This means that any operator $f \in \widehat{\mathcal{A}}_{MN}$ can be expanded into a base of operators transforming according to (27).

We remark, that the description of topologically nontrivial field configurations with $2k \ne 0$ in the noncommutative setting needs two algebras $\widehat{\mathcal{A}}_{MM}$ and $\widehat{\mathcal{A}}_{NN}$ with $2k = M - N$. A noncommutative analog of a complex scalar field ϕ with fixed winding number k we identify as element of $\widehat{\mathcal{A}}_{MN}$. For the action we take

$$S_{MN}\,[\phi] = \frac{2}{M+N+2}\,\mathrm{Tr}_N\left(\phi^+\,(K_+ K_- + K_- K_+)\,\phi + V(\phi^+, \phi)\right), \tag{28}$$

where the operators $K_\pm$ are defined by

$$K_+ \phi = i \, \epsilon_{\alpha\beta} \, A^+_\beta \, [\phi, A^+_\alpha] \quad , \quad K_- \phi = i \, \epsilon_{\alpha\beta} \, [A_\alpha, \phi] \, A_\beta \tag{29}$$

and $2K_0 = \mathrm{ad}\,(A^+_\alpha \, A_\alpha)$ takes on the constant value $2k$. The order of operators in (29) is essential, so that they act like in the commutative case. Moreover, putting $2J = M+N$ and keeping k fixed, we recover the commutative action in the limit $J \to \infty$. We conclude, that we obtain this way a fully $su(2)$ symmetric, that means rotation symmetric model, which is described by a finite number of modes. In a certain sense, we work on a noncommutative finite "lattice".

4. The Spinor Field

The spinor bundle over S_2 is a standard part of any textbook of quantum mechanics. Also the spectrum of the Dirac operator and all eigenfunctions (the spinorial harmonics) are known, and we are therefore very brief. We may start from the spinor bundle over $\mathbf{R}^3$ with sections being two component spinorial wave functions (ψ_+, ψ_-). The action of $su(2)$ is described by the generators $J_i = L_i + \frac{\sigma_i}{2}$. View $\mathbf{R}^3$ as $\mathbf{R}_+ \times S^2$. A simple exercise allows to express the flat Dirac operator in three dimensions in terms of spherical coordinates. This way we obtain the Dirac operator corresponding to the round metric on S^2 in terms of the $su(2)$ generators as follows [3]

$$D = \frac{1}{R} \, (\vec{\sigma} \cdot \vec{L} + 1) \, , \tag{30}$$

D is self-adjoint with respect to the scalar product

$$\langle \psi \mid \varphi \rangle = \int \frac{d^3 \, X}{2\pi R} \, \delta(\vec{X}^2 - R^2) \, (\psi^+_+ \, \varphi_+ + \psi^+_- \, \varphi_-) \, , \tag{31}$$

where $\psi = (\psi_+, \psi_-)$ and $\varphi = (\varphi_+, \varphi_-)$ belong to the spinor bundle $S_{\mathcal{A}_\infty}$. From the point of view of transformation properties under $su(2)$, $S_{\mathcal{A}_\infty}$ transforms according to the tensor product of the spin 1/2 representation [1/2] times representations of $\mathcal{A}_\infty$:

$$S_{\mathcal{A}_\infty} = 2 \cdot \left(\left[\frac{1}{2}\right] \oplus \left[\frac{3}{2}\right] \oplus \left[\frac{5}{2}\right] \oplus \cdots \right) . \tag{32}$$

All half-integer spin representations occur exactly twice. The doubling corresponds to left and right chiral spinor bundles. It is equally easy to work out the Dirac operator D_k on spinors with components ψ_α, which are sections belonging to the set $\mathcal{A}_k$ (16). They become

$$D_k = \frac{1}{R} \, (\vec{\sigma} \cdot \vec{J} + 1 + k \, \vec{\sigma} \cdot \vec{x}) \, , \tag{33}$$

where $\vec{\sigma}\cdot\vec{x}/R = \Gamma$ equals the chirality operator. The winding number k equals the monopole charge. The spectrum of D_k from (33) is given by

$$E_{k,j} = \pm\sqrt{\left(j+\frac{1}{2}\right)^2 - k^2}\,, \tag{34}$$

where $j = |k|+1/2, |k|+3/2, \ldots$ corresponds to the non-zero-eigenvalue modes. $j = |k|-\frac{1}{2}$ gives zero modes and is admissible only for $k \neq 0$. For $k > 0$ negative chirality zero modes exist, while for $k < 0$ positive chirality zero modes appear.

We next look for a noncommutative analog of the spinor bundle. A counting of degrees of freedom of vector spaces entering in (32) shows that the naive approach of considering the elements of $\widehat{\mathcal{A}}_{MN}$ as the spinor components does not work. The noncommutative sphere described in chapter two emerged from the quantization of the algebra of scalar functions on the ordinary sphere. It was therefore natural to treat bosons and fermions simultaneously and to study the supersymmetric extension of the above scheme. We found that a consistant truncation (deformation quantization) of the ring of scalar superfields is possible. The fermions are then identified as components of these superfields. In the topologically non-trivial case, we have obtained the quantization [11] by deforming the $N = 1$ superextension of the Hopf fibration described in chapter three.

As the Hopf superfibration [11] we denote the mapping from $\mathbf{C}^{2,1}$ to $\mathbf{R}^{3,2}$ given by

$$\xi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ a \end{pmatrix} \leftrightarrow (X_i;\theta_+,\theta_-) \quad , \quad i = 1,2,3 \tag{35}$$

$$X_i = \xi^+ R_i\, \xi \qquad , \qquad \theta_\pm = \xi^+ F_\pm\, \xi$$

where χ_i are two complex even parameters, while a is a complex odd parameter. $R_3, R_\pm = R_1 \pm iR_2$ and $F_\pm$ are 3×3 matrices given as

$$F_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad F_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R_+ = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,\ R_- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,\ R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

They generate the $\mathrm{osp}\,(2,1)$ superalgebra [3]. F_+ and F_- are the odd generators. Their brackets gives the three even ones:

$$\{F_+, F_-\} = -R_3 \quad , \quad \{F_\pm, F_\pm\} = \pm R_\pm\,. \tag{37}$$

$R_\pm$ and R_- obey $su(2)$ algebra relations among themselves and even and odd relations

$$[R_3, F_\pm] = \pm\frac{1}{2}\, F_\pm \quad , \quad [R_\pm, F_\pm] = 0 \quad , \quad [R_\pm, F_\mp] = F_\pm \tag{38}$$

show that $F_\pm$ transforms like a $su(2)$ spinor and the superalgebra closes.

A superfunction is now defined as a linear combination of monomials in χ^+_α, χ_α and a^+, a. To such a monomial we assign a topological charge $2k$ and define the set of superfunction

$$\begin{aligned}(s\,\mathcal{A}_\infty)_k = \Big\{ f = \sum a_{m_1 m_2 n_1 n_2 \mu\nu}\, (\chi_1^+)^{m_1}\, (\chi_2^+)^{m_2}\, (\chi_1)^{n_1}\, (\chi_2)^{n_2}\, a^{+\mu}\, a^\nu \mid \\ 2k = m_1 + m_2 - n_1 - n_2 + \mu - \nu \Big\}\end{aligned} \tag{39}$$

$\phi \in (s\,\mathcal{A}_\infty)_k$ can be expanded as

$$\phi = \phi_0(\chi^+, \chi) + f(\chi^+, \chi)\, a + g(\chi^+, \chi)\, a^+ + F(\chi^+, \chi)\, a^+\, a\,, \tag{40}$$

where $\phi_0, F \in (\mathcal{A}_\infty)_k$, $f \in (\mathcal{A}_\infty)_{k+1/2}$ and $g \in (\mathcal{A}_\infty)_{k-1/2}$. Much as before there exist an involutive gradation. $(s\,\mathcal{A}_\infty)_0$ is a superalgebra with respect to the supercommutative product of parameters ξ^+ and ζ, $(s\,\mathcal{A}_\infty)_k$ are $(s\,\mathcal{A}_\infty)_0$-bimodules.

The differential operators generating the osp $(2,1)$ algebra in $(s\,\mathcal{A}_\infty)_k$ are given by

$$\begin{aligned} J_k &= \frac{i}{2}\, (\xi^+_\alpha\, R^k_{\alpha\beta}\, \partial_{\xi^+_\beta} - \xi_\alpha\, R^{*k}_{\alpha\beta}\, \partial_{\xi_\beta}) \\ V_\pm &= \frac{i}{2}\, (\xi^+_\alpha\, F^\pm_{\alpha\beta}\, \partial_{\xi^+_\beta} + \xi_\alpha\, (F^\pm_{\alpha\beta})^*\, \partial_{\xi_\beta})\,. \end{aligned} \tag{41}$$

We deduce by explicite calculation, that $C(X,\theta) = \sum_i X_i^2 + \frac{1}{2}\,(\theta_+\,\theta_- - \theta_-\,\theta_+)$ is an invariant superfunction from $(s\,\mathcal{A}_\infty)_0$:

$$J_i\, C(X,\theta) = V_\pm\, C(X,\theta) = 0\,.$$

We can therefore define the supersphere in $\mathbf{R}^{3,2}$ by the condition that $C(X,\theta) = R^2$. Note that the superfunctions are understood to be restricted to the supersphere. The Berezin integral times the standard measure on the sphere allows to introduce an inner product over the supersphere.

Before turning to the non commutative truncation, we observe that irreducible representations of osp $(2,1)$ consist of pairs of $su(2)$ irreducible representations $[\tilde{j}] = [j] \oplus [j - \frac{1}{2}]$, where j is either an integer or a half-integer. The generators $X_i, \theta_\pm \in (s\,\mathcal{A}_\infty)_0$ form a superspin 1 irreducible representation of the osp $(2,1)$ algebra under the action of the vector

fields (41) and higher powers of X_i and $\theta_\pm$ can be arranged into irreducible supermultiplets of higher superspins. From the point of view of $su(2)$ representations, the algebra $(s\,\mathcal{A}_\infty)_0$ of superfields consists of two copies of $\mathcal{A}_\infty$ and the spinor bundle $\left[\frac{1}{2}\right]\otimes\mathcal{A}_\infty$. This gives the full decomposition of $(s\,\mathcal{A}_\infty)_0$ into irreducible representations of osp $(2,1)$ as an infinite direct sum

$$(s\,\mathcal{A}_\infty)_0 = \widetilde{[0]} \oplus \widetilde{\left[\frac{1}{2}\right]} \oplus \widetilde{[1]} \oplus \widetilde{\left[\frac{3}{2}\right]} \oplus \cdots . \tag{42}$$

It is instructive to insert into (42) the spin representations and count the degrees of freedom. Whenever you truncate they sum up to a complete square.

Thus, we proceed as in chapter two and define the truncations of $(s\,\mathcal{A}_\infty)_0$ as the family of noncommutative superspheres $(s\,\mathcal{A}_j)_0$, being given by the sum of irreducible representations of osp $(2,1)$

$$(s\,\mathcal{A}_j)_0 = \widetilde{[0]} \oplus \widetilde{\left[\frac{1}{2}\right]} \oplus \widetilde{[1]} \oplus \widetilde{\left[\frac{3}{2}\right]} \oplus \ldots \oplus \widetilde{[j]} \quad , \quad j \in Z \tag{43}$$

together with an associative product and an inner product such that in the limit $j\to\infty$ the standard product of $(s\,\mathcal{A}_\infty)_0$ is obtained. We can represent the algebra $(s\,\mathcal{A}_j)_0$ in the Fock space $(s\,\mathcal{F})$ of bosonic and fermionic degrees of freedom. We replace

$$(\chi^+_\alpha,\chi_\alpha) \to \left(\frac{1}{\sqrt{s\hat{N}}}\,A^+_\alpha, A_\alpha\frac{1}{\sqrt{s\hat{N}}}\right),\ (a^+,a) \to \left(\frac{1}{\sqrt{s\hat{N}}}\,b^+, b\frac{1}{\sqrt{s\hat{N}}}\right),\ s\hat{N} = \hat{N} + b^+ b, \tag{44}$$

where b^+, b are fermionic creation and annihilation operators and $\hat{N}$ was defined in (21). $(s\,\mathcal{F})$ is spanned by vectors

$$|n_1,n_2;\nu\rangle = \frac{1}{\sqrt{n_1!\,n_2!}}\,(A^+_1)^{n_1}\,(A^+_2)^{n_2}\,(b^+)^\nu\,|0\rangle \tag{45}$$

with $n_i\geq 0$, $\nu = 0,1$, and the supervacuum is defined by $A_\alpha\,|0\rangle = b\,|0\rangle = 0$. The operators

$$\widehat{X}_j = \frac{1}{2}\,A^+_\alpha\,\sigma^j_{\alpha\beta}\,A_\beta \quad , \quad \widehat{\theta}_+ = \frac{-1}{\sqrt{2}}\,(A^+_1\,b + A_2\,b^+) \quad , \quad \widehat{\theta}_- = \frac{1}{\sqrt{2}}\,(-A^+_2\,b + A_1\,b^+) \tag{46}$$

satisfy in $(s\,\mathcal{F})$ the osp $(2,1)$ superalgebra commutation relations (37) and (38). The "superadjoint" action of the form

$$\widehat{J}_k\,\phi = [\widehat{X}_k,\phi] \quad , \quad \widehat{V}_\pm = [\widehat{\theta}_\pm,\phi]_g\,, \tag{47}$$

where $[\cdot,\cdot]_g$ denotes the graded commutator, defines a reducible representation of osp $(2,1)$. As

$$(s\,\mathcal{F})_N = \{|n_1,n_2;\nu\rangle \in (s\,\mathcal{F}) \mid n_1+n_2+\nu = N\} \tag{48}$$

we denote a $(2N+1)$-dimensional subspace, which may be decomposed into a subspace with N bosons and no fermions and a subspace with $(N-1)$ bosons and one fermion. $(s\,\mathcal{F})_N$ is the representation space of an irreducible representation of $\mathrm{osp}\,(2,1)$ where the Casimir operator

$$C = \widehat{X}_3^2 + \frac{1}{2}\,\{\widehat{X}_+, \widehat{X}_-\} + \frac{1}{2}\,[\widehat{\theta}_+, \widehat{\theta}_-] \tag{49}$$

takes on the value $N(N+1)/4$. As $(s\,\widehat{\mathcal{A}})_{MN}$ we denote the space of linear mappings from $(s\,\mathcal{F})_N$ to $(s\,\mathcal{F})_M$ spanned by operator monomials of the form (39) with $m_1+m_2+\mu \le M$, $n_1+n_2+\nu \le N$ and $m_1+m_2+\mu-n_1-n_2-\nu = M-N$. These operators are represented by $(2N+1)\times(2M+1)$ matrices. In $(s\,\widehat{\mathcal{A}})_{MN}$ an inner product is defined by the supertrace $\langle\phi/\psi\rangle_{MN} = \ s\,\mathrm{Tr}\,(\phi^+\,\psi)$ in the space of linear operators from $(s\,\mathcal{F})_N$ to $(s\,\mathcal{F})_N$.

The spinor field we identify in the supercommutative case as the odd part of the superfield ϕ of equation (40):

$$\psi = f\,(\chi^+, \chi)\,a + g\,(\chi^+, \chi)\,a^+\,, \tag{50}$$

f and g correspond to chirality eigenmodes.

Γ maps ψ into $\Gamma\,\psi = f\,a - g\,a^+$. The Dirac operator maps from $(s\,\mathcal{A})_k$ to $(s\,\mathcal{A})_k$ as

$$D\,\psi = (K_+\,g)\,a + (K_-\,f)\,a^+\,, \tag{51}$$

and anticommutes with $\Gamma : \{D, \Gamma\} = 0$.

Next we return to the truncated situation. As $\widehat{S}_k$, $k \in \frac{1}{2}\,Z$, we denote the set of odd elements from $(s\,\widehat{\mathcal{A}})_k$, with $f \in \widehat{\mathcal{A}}_{k+1/2}$ and $g \in \widehat{\mathcal{A}}_{k-1/2}$, where $(s\,\widehat{\mathcal{A}})_k$ are $(s\,\widehat{\mathcal{A}})_0$-modules generated by the operators (44) before the restriction to a particular $(s\,\mathcal{F})_N$. Note that $\widehat{S}_k$ are $\widetilde{\mathcal{A}}_0$-bimodules but not $(s\,\widehat{\mathcal{A}})_0$-bimodules. Equation (50) with non-commutative $\hat{\chi}^+, \hat{\chi}, \hat{a}$ given by (44) again induces a decomposition of the spinor space $\widehat{S}_k$ into the direct sum of positive and negative chirality contribution. The noncommutative analog of the chirality operator $\widehat{\Gamma}$ acts as

$$\widehat{\Gamma}\,\psi = -[b^+\,b, \psi] \tag{52}$$

and the free Dirac operator on $\widehat{S}_k$ becomes

$$\hat{D}\,\psi = (K_+\,g)\,\hat{a} + (K_-\,f)\,\hat{a}^+\,, \tag{53}$$

with $\{\widehat{\Gamma}, \widehat{D}\} = 0$.

As $\widehat{S}_{MN}$ we denote odd mappings from $(s\,\widehat{\mathcal{A}})_{MN}$ which are of the form

$$\psi = f(\hat{\chi}^+, \hat{\chi})\,\hat{a} + g\,(\hat{\chi}^+, \hat{\chi})\,\hat{a}^+ \tag{54}$$

where $f \in \widehat{\mathcal{A}}_{M,N-1}$ and $g \in \widehat{\mathcal{A}}_{M-1,N}$. This means that f and g can be expanded in terms of operators transforming according to the representations

$$\begin{aligned} f: \left[\frac{M}{2}\right] \otimes \left[\frac{N-1}{2}\right] &= \left[k+\frac{1}{2}\right] \oplus \cdots \oplus \left[J-\frac{1}{2}\right] \\ g: \left[\frac{M-1}{2}\right] \otimes \left[\frac{N}{2}\right] &= \left[k-\frac{1}{2}\right] \oplus \cdots \oplus \left[J-\frac{1}{2}\right], \end{aligned} \tag{55}$$

for $k = \frac{1}{2}(M-N) \neq 0$ and $J = \frac{1}{2}(M+N)$. In terms of eigen-operators of $\vec{J}^2$ and J_3 to eigenvalues j and m, f can be expanded in operators with $j = \left|k+\frac{1}{2}\right|, \ldots J-\frac{1}{2}$, $|m| \leq j$, and g into operators with $j = \left|k-\frac{1}{2}\right|, \ldots J-\frac{1}{2}$, $|m| \leq j$. The admissible values for j are: $j = |k|-\frac{1}{2}$, $|k|+\frac{1}{2}, \ldots, J-\frac{1}{2}$, but the first value $j = |k|-\frac{1}{2}$ can occur only for $k \neq 0$. For $k > 0$ zero modes occur with negative chirality $(f = 0, g \neq 0)$, for $k < 0$ vice versa they occur with positive chirality $(f \neq 0, g = 0)$. The number of zero modes of the Dirac operator is always $|M-N|$. We note that the spectrum of $\widehat{D}$ on $\widehat{S}_{MN}$ is identical to that of D_k given by (33), except that it is cut off at $j = \frac{M+N-1}{2}$.

The action of a selfinteracting spinor field $\psi \in \widehat{S}_{MN}$ with fixed winding number is defined by

$$S_{MN}[\psi^+, \psi] = s \, \mathrm{Tr}_N \left(\psi^+ D \, \psi + W(\psi^+, \psi)\right). \tag{56}$$

This action is invariant with respect to the following symmetries:

(i) rotations of the sphere,

(ii) chiral transformations $\psi \to e^{i\alpha\widehat{\Gamma}} \, \psi$, $\psi^+ \to \psi^+ \, e^{i\alpha\widehat{\Gamma}}$,

as long as the interaction term in (56) is invariant. Expanding ψ in terms of eigen-operators of $\widehat{D}$, a functional integral is given by integrating over the independent Grassmann expansion coefficients. Chiral transformations leave invariant this measure $[D \, \psi^+ \, D \, \psi]_{MN}$ expect for the zero mode contributions. For $k \neq 0$, chiral symmetry is violated on the quantum level and

$$[D \, \psi^+ \, D \, \psi]_{MN} \to e^{-i\alpha k} \, [D \, \psi^+ \, D \, \psi]_{MN} \tag{57}$$

under chiral transformations.

Our main tool was to provide a cut-off procedure for simple models, which respects the symmetries one started with. The supersymmetry approach allows to describe chiral spinors. The description in terms of rectangular matrices allows to approximate also topological nontrivial configurations. A study of coupling to gauge fields has been done too [8].

Many questions remain, especially: Study of the limit $N \to \infty$; different manifolds; more dimensional situations. There may be also an interesting connection of our formalism

with the description of the bound states of strings and p-branes [12] where the space-time coordinates also enter in the formalism as non-commuting matrices.

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